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FUZZY PROBABILISTIC METHOD AND ITS APPLICATION FOR THE SAFETY ASSESSMENT OF STRUCTURES

Bernd Möller, Wolfgang Graf, Michael Beer, Jan-Uwe Sickert

Institute of Structural Analysis
Dresden University of Technology, Dresden, Germany
e-mail: statik@rcs.urz.tu-dresden.de

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Abstract. *The safety of structures may only be realistically assessed provided all input data are appropriately described and a realistic computational model is implemented. Input and model parameters are often only available in the form of uncertain parameters. Statistical methods are only suitable for describing uncertainty to a limited extent, however; the prognoses given by stochastic safety models are thus open to criticism. In this paper a new method of modeling uncertainty is presented, based on the theory of fuzzy random variables. Uncertain parameters, which partly exhibit random properties, but may not be modeled as random variables without an element of doubt, are described using fuzzy probability distributions. These enter the developed fuzzy probabilistic safety assessment as fuzzy probabilistic basic variables (data uncertainty). In contrast to probabilistic concepts, this new safety concept treats uncertain input and model parameters as fuzzy random variables, random variables and fuzzy variables. Random variables are additionally modeled as probabilistic basic variables (data uncertainty); fuzzy variables (model uncertainty) define the fuzzy limit state surface. Using the special extension of the First Order Reliability Method (FORM), namely the Fuzzy First Order Reliability Method (FFORM), the fuzzy reliability index is computed by α-level optimization. This is compared with required values. Expert estimates are taken into consideration for assessing the fuzzy safety level. The fuzzy probabilistic safety assessment is demonstrated by way of an example.*

1 Introduction

The consideration of uncertainty in both data and models is an important prerequisite for realistic structural analysis and proper safety assessment.

According to [1], uncertainty is the gradual assessment of the truth content of a postulation, e.g. in relation to the occurrence of a defined event. All non-deterministic parameters are characterized by uncertainty; these are referred to as uncertain parameters.

The classification and description of uncertainty may be carried out according to different criteria. In this paper the concept suggested in [2] is adopted; uncertainty is classified in such a way that a mathematically-founded and realistic description is ensured in the structural analysis and the safety assessment. This classification is shown in Fig. 1 according to type and characteristics.

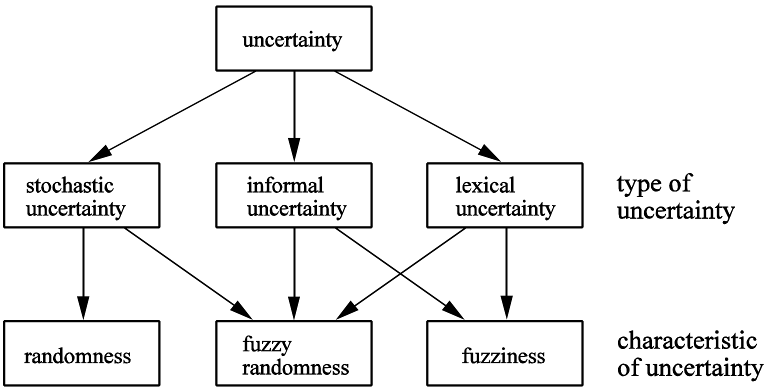

Fig. 1 Classification of uncertainty according to type and characteristics

Whereas the type of the uncertainty indicates the cause of its manifestation, the characteristics of the uncertainty are described by the mathematical properties, randomness, fuzziness and fuzzy randomness. The uncertainty characteristics depend on the type of uncertainty and the information content of the uncertain parameters.

Another classification distinguishes between data uncertainty and model uncertainty. This is linked to the definition of the model in any particular case. In the safety assessment the limit state surface is solely determined by the computational model. Model uncertainty in this case refers to uncertainty in the limit state surface; the uncertainty in the basic variables is referred to as data uncertainty.

Different methods are available for mathematically describing and quantifying uncertainty. These include e.g. the probability theory [3], the interval algebra [4], convex modeling [5], fuzzy set theory [1] and the theory of fuzzy random variables [6].

Conventional methods for structural analysis and safety assessment only permit the modeling of the uncertainty of structural parameters and structural models to a limited extent. The information content of the uncertainty of input parameters and models is often inadequately described and accounted for, or in some cases, ignored altogether. The possibilities available for taking uncertainty into account are limited.

Probabilistic concepts [7] presuppose sufficient information for determining stochastic input parameters, such as, e.g. expected values, variances, quantile values and probability distribution functions. The quality of the input information must be statistically assured by a sufficiently large set of samples. Probabilistic methods are only able to account for uncertainty with the characteristic randomness (stochastic uncertainty). Inaccuracies, unreliable data, or uncertainty which cannot be described or insufficiently described statistically can thus only be accounted for approximately. In view of the latter, probabilistic methods may only be applied to a limited extent.

Alternative methods based on fuzzy set theory and the theory of fuzzy random variables have only been applied in past few years. For both structural analysis and safety assessment, algorithms have been developed which take into account non-stochastic uncertainty [8]. In [9] a method is presented for the numerical simulation of the structural behavior of systems in which fuzzy structural parameters and fuzzy model parameters occur. This fuzzy structural analysis is based on α-level optimization. A possibilistic safety concept for assessing structural reliability using fuzzy variables is presented in [10]. Initial ideas relating to the fuzzy probabilistic method dealt with here are outlined in [11].

Uncertainty with the characteristic fuzzy randomness is described, quantified and processed on the basis of the theory of fuzzy random variables. This includes, as also in the case of fuzziness, both objective and subjective information. The theory of fuzzy random variables permits the modeling of uncertain structural parameters, which partly exhibit randomness, but which cannot be described using random variables without an element of doubt. The randomness is "disturbed" by a fuzziness component. The reasons for the existence of fuzzy randomness might be:

1) Although samples are available for a structural parameter, these are only limited in number. No further information exists concerning the statistical properties of the universe.

2) The statistical data material possesses fuzziness, i.e.,
   - the sample elements are of doubtful accuracy,
   - or they were obtained under unknown or non-constant reproduction conditions.

In order to take account of uncertainty with the characteristic fuzzy randomness the method of fuzzy probabilistic safety assessment is developed. This comprehensive safety concept is formulated as a further development of introduced probabilistic approaches.

The following sub-problems are presented in this paper:

- Extension of the theory of fuzzy random variables and application of α-discretization
- Formulation of fuzzy probability distribution functions
- Determination and assessment of the fuzzy safety level.

2 Fuzzy random variables

The underlying concepts and definitions relating to the theory of fuzzy random variables are dealt with in [6], [12], [13] and [14]. The mathematical method is extended in terms of measure and set theory in such a way that α-discretization can be applied to fuzzy random variables [2]. By this means, the prerequisites are established for the application of α-level optimization [9].

The definitions and properties of fuzzy random variables, which are relevant to the formulation of fuzzy reliability theory, are developed by extending the axiomatic probability concept after KOLMOGOROW. The probability space [$\underline{\mathbf{X}}$; S; P] is thereby extended by the dimension of fuzziness; the uncertain measure probability remains defined over the n-dimensional EUCLIDian space $\mathbb{R}^n$.

2.1 Definition

If the space of the random elementary events, as in probabilistics, is described by Ω, a fuzzy random variable on the fundamental set $\underline{\mathbf{X}} = \mathbb{R}^n$ may be defined as follows.

A *fuzzy random variable* $\tilde{\underline{X}}$ is the fuzzy result of the uncertain mapping

$$\Omega \rightleftarrows \mathbf{F}(\mathbb{R}^n) \tag{1}$$

where $\mathbf{F}(\mathbb{R}^n)$ is the set of all fuzzy numbers in $\mathbb{R}^n$.

An ordered n-tupel of fuzzy numbers $\tilde{x}_i$ (with the membership functions $\mu(x_i)$) is assigned to each (crisp) elementary event $\omega \in \Omega$. The n-tupel $\tilde{\underline{x}} = (\tilde{x}_1; ...; \tilde{x}_n) \subseteq \underline{\mathbf{X}}$ is a realization of the fuzzy random variable $\tilde{\underline{X}}$. Several realizations for a one-dimensional fuzzy random variable are presented in Fig. 2.

If the realization $\underline{x}$ of an ordinary random variable $\underline{X}$ as well as the fuzzy realization $\tilde{\underline{x}}$ of a fuzzy random variable $\tilde{\underline{X}}$ may be assigned to an elementary event ω, and if $\underline{x} \in \tilde{\underline{x}}$ holds, this means that $\underline{x}$ is contained in $\tilde{\underline{x}}$. If, for all elementary events $\omega \in \Omega$, the $\underline{x}$ are contained in the $\tilde{\underline{x}}$, the $\underline{x}$ then constitute an *original* $\underline{X}$ of the fuzzy random variable $\tilde{\underline{X}}$. The original $\underline{X}$ is referred to as *completely* contained in $\tilde{\underline{X}}$. Each ordinary random variable $\underline{X}$ (without fuzziness) on $\underline{\mathbf{X}}$ which is completely contained in $\tilde{\underline{X}}$ is thus an original of $\tilde{\underline{X}}$, i.e., in the Ω-direction each original $\underline{X}$ must be consistent with the fuzziness of $\tilde{\underline{X}}$. This means that the fuzzy random variable $\tilde{\underline{X}}$ is the *fuzzy set of all possible originals* $\underline{X}$ contained in $\tilde{\underline{X}}$. Fuzzy random variables may be continuous or discrete with regard to both their randomness and fuzziness.

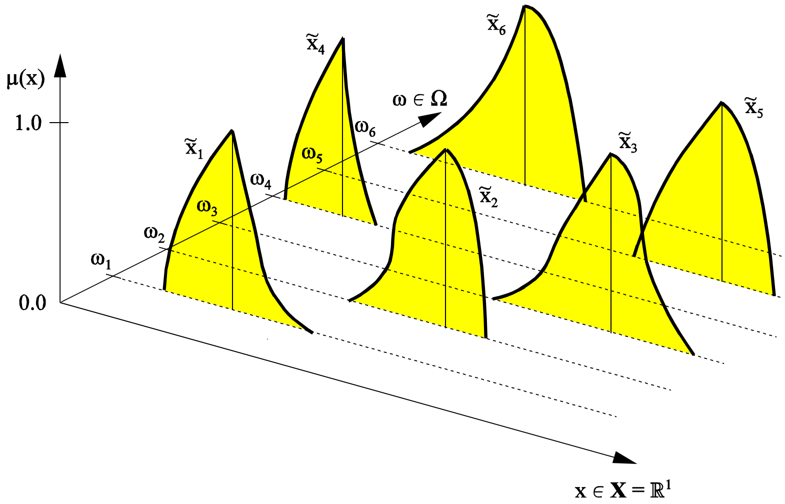

Fig. 2 Realizations of a one-dimensional fuzzy random variable

Each fuzzy random variable $\underline{\tilde{X}}$ contains at least one ordinary random variable $\underline{X}$ as an original of $\underline{\tilde{X}}$. It thus follows that each fuzzy random variable $\underline{\tilde{X}}$, which only possesses precisely one original, is an ordinary random variable $\underline{X}$. The description of fuzzy random variables by means of their originals ensures that ordinary random variables are contained in fuzzy random variables as a special case. As each realization $\underline{\tilde{x}}$ of the fuzzy random variable $\underline{\tilde{X}}$ according to Eqn. (1) is a fuzzy number, the special case is an ordinary random variable uniquely defined by the mean values of the fuzzy realizations $\underline{\tilde{x}}$ (membership level $\mu = 1$). By this means it is possible to take account of random variables and fuzzy random variables simultaneously as probabilistic and fuzzy probabilistic basic variables.

2.2 Probability measure for fuzzy random variables

The probability measure is derived from the following treatment. Given are the realizations $\underline{\tilde{x}}$ of the fuzzy random variable $\underline{\tilde{X}}$; the $\underline{\tilde{x}}$ are fuzzy sets on the fundamental set $\underline{\mathbf{X}}$. Moreover, the system of sets $\mathsf{M}(\underline{\mathbf{X}})$, whose crisp elements $\underline{A}_i$ are treated as events E_i, is defined on $\underline{\mathbf{X}}$. The event E_i is considered to have occurred when $\underline{\tilde{X}} = \underline{\tilde{x}} \in \underline{A}_i$. Due to the fuzziness of $\underline{\tilde{x}}$ the event E_i also possesses fuzziness; this becomes the fuzzy event $\tilde{E}_i$.

The fuzzy probability $\tilde{P}(\underline{A}_i)$ *is the set of all probabilities* $P(\underline{\tilde{X}} \in \underline{A}_i)$ *with the corresponding membership values* $\mu(P(\underline{\tilde{X}} \in \underline{A}_i))$*, which takes into account all states of the (also partial) occurrence of* $\underline{\tilde{X}} \in \underline{A}_i$.

If the underlying system of sets $\mathsf{M}(\underline{\mathbf{X}})$ satisfies the demands posed on a σ-algebra $\mathsf{S}(\underline{\mathbf{X}})$ and if the postulation $\underline{\tilde{X}} = \underline{\tilde{x}} \in \underline{A}_i$ (for an arbitrary assessment of $\underline{\tilde{x}} \in \underline{A}_i$ according to fuzzy set theory) is replaced by the fuzzy set $\underline{\tilde{A}}_i$, then the system of all fuzzy sets $\underline{\tilde{A}}_i$ forms a fuzzy σ-algebra $\tilde{\mathsf{S}}(\underline{\mathbf{X}})$ over $\mathbb{R}^n$. As this fuzzy σ-algebra is no longer a BOOLEan algebra the KOLMOGOROWian axioms must not be applied.

By means of α-discretization the determination of $\tilde{P}(\underline{A}_i)$ reduces to the determination of probabilities in the ordinary probability space [$\underline{\mathbf{X}}$; S; P]. If the fuzzy set $\underline{\tilde{A}}_i$ is subdivided into the crisp sets $\underline{A}_{i,\alpha}$ with $\alpha \in [0; 1]$, the fuzzy σ-algebra $\tilde{S}(\underline{\mathbf{X}})$ reduces to individual (BOOLEan) σ-algebras $S_\alpha(\underline{\mathbf{X}})$. The $\underline{A}_{i,\alpha}$ may now be found by evaluating $\underline{\tilde{x}} \in \underline{A}_i$ for each α-level. For this purpose the fuzzy random variable $\underline{\tilde{X}}$ is also subdivided into α-level sets

$$\underline{X}_\alpha = \{\underline{X} \mid \mu(\underline{X}) \geq \alpha\} \tag{2}$$

The $\underline{X}_\alpha$ are random-dependent (crisp) sets. In the one-dimensional case a bounded random interval $[X_{\alpha\,l}; X_{\alpha\,r}]$ is obtained. All elements of $\underline{X}_\alpha$ or $[X_{\alpha\,l}; X_{\alpha\,r}]$ are originals of $\underline{\tilde{X}}$. The probability that the element $\underline{X}$ of the α-level set $\underline{X}_\alpha$ is also an element of $\underline{A}_i$ may now be stated on each α-level using the uncertain measure probability (see Fig. 3).

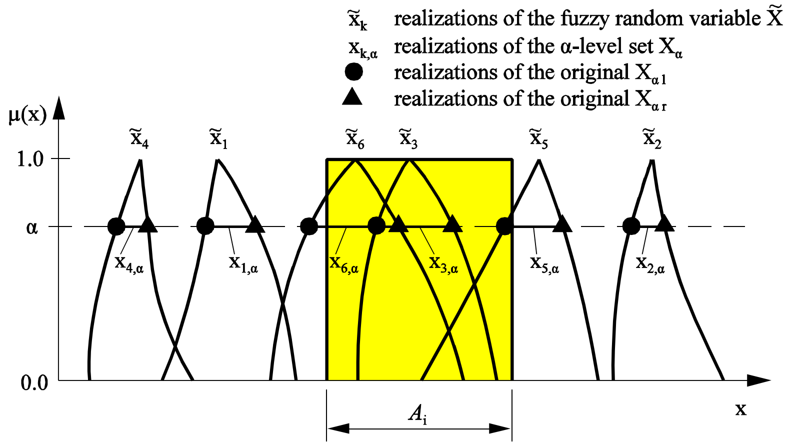

Fig. 3 Crisp set A_i and realizations $\tilde{x}$ of the fuzzy random variable $\underline{\tilde{X}}$ by applying α-discretization

The probability with which elements of a random, not yet observed, realization of the fuzzy random variable $\underline{\tilde{X}}$ and elements of a crisp set $\underline{A}_i$ (defined on the fundamental set $\underline{\mathbf{X}}$) coincide is referred to as the *fuzzy probability* $\tilde{P}(\underline{A}_i)$ of $\underline{A}_i$

$$\tilde{P}(\underline{A}_i) = \{(P_\alpha(\underline{A}_i); \mu(P_\alpha(\underline{A}_i))) \mid P_\alpha(\underline{A}_i) = [P_{\alpha\,l}(\underline{A}_i); P_{\alpha\,r}(\underline{A}_i)]; \mu(P_\alpha(\underline{A}_i)) = \alpha \,\forall\, \alpha \in (0; 1]\} \tag{3}$$

The corresponding measure space is referred to as the fuzzy probability space $[\underline{\mathbf{X}}; \tilde{S}; \tilde{P}]$.

The right-hand side of Eqn. (3) is evaluated for each α-level with the aid of the (ordinary) measure probability. The boundaries $P_{\alpha\,l}(\underline{A}_i)$ and $P_{\alpha\,r}(\underline{A}_i)$ of the α-level sets $P_\alpha(\underline{A}_i)$ are obtained by evaluating the events

E_1: "$\underline{X}_\alpha$ is contained in $\underline{A}_i$: $\underline{X}_\alpha \subseteq \underline{A}_i$"

and

E_2: "$\underline{X}_\alpha$ and $\underline{A}_i$ possess at least one common element: $\underline{X}_\alpha \cap \underline{A}_i \neq \emptyset$".

These two events are "extreme" interpretations of the postulation $\underline{\tilde{X}} \in \underline{A}_i$ for formulating the $\tilde{\underline{A}}_i$ of the fuzzy σ-algebra $\tilde{S}(\underline{X})$. The event E_1 yields the least probability $P_{\alpha\,l}(\underline{A}_i)$ whereas the event E_2 yields the highest probability $P_{\alpha\,r}(\underline{A}_i)$, see Fig. 4. The events E_1 and E_2 characterize bounds with regard to a partial occurrence of $\underline{\tilde{X}} \in \underline{A}_i$. The following holds

$$P_{\alpha\,l}(\underline{A}_i) = P(\underline{X}_\alpha \subseteq \underline{A}_i) \tag{4}$$

$$P_{\alpha\,r}(\underline{A}_i) = P(\underline{X}_\alpha \cap \underline{A}_i \neq \varnothing) \tag{5}$$

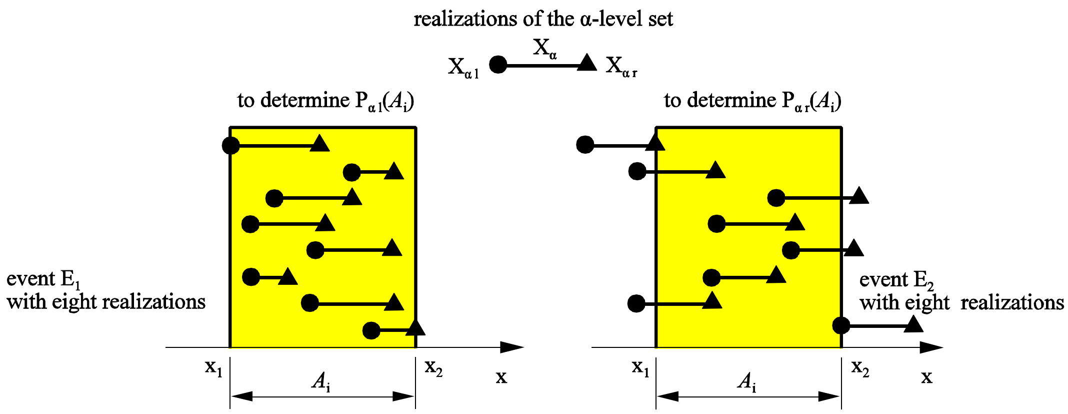

Fig. 4 Events for determining $P_{\alpha\,l}(A_i)$ and $P_{\alpha\,r}(A_i)$ for a one-dimensional fuzzy random variable

As *all* elements of $\underline{X}_\alpha$ are originals of $\underline{\tilde{X}}$, Eqns. (4) and (5) also yield bounds for the probability of the originals. The intervals $[P_{\alpha\,l}(\underline{A}_i);\, P_{\alpha\,r}(\underline{A}_i)]$ contain the probabilities of all possible states describing a partial occurrence of $\underline{\tilde{X}} \in \underline{A}_i$.

For the special case of ordinary random variables $\underline{X}$, both $\underline{X}_\alpha \cap \underline{A}_i \neq \varnothing$ as well as $\underline{X}_\alpha \subseteq \underline{A}_i$ reduce to $\underline{X} \in \underline{A}_i$, i.e., $P_{\alpha\,l}(\underline{A}_i)$ and $P_{\alpha\,r}(\underline{A}_i)$ coincide; the event $\underline{X} \in \underline{A}_i$ cannot occur partially.

The partial occurrence of $\underline{\tilde{X}} \in \underline{A}_i$ is induced by the fuzziness of $\underline{\tilde{X}}$. The evaluation of all states, for which $\underline{X}_\alpha \cap \underline{A}_i \neq \varnothing$ is satisfied, but for which $\underline{X}_\alpha \subseteq \underline{A}_i$ does not apply, yields the *probability shadow* $\tilde{P}_S(\underline{A}_i)$, which exclusively contains the fuzziness of the event $\underline{\tilde{X}} \in \underline{A}_i$

$$\tilde{P}_S(\underline{A}_i) = \{(P_{S,\alpha}(\underline{A}_i); \mu(P_{S,\alpha}(\underline{A}_i))) \mid P_{S,\alpha}(\underline{A}_i) = [0; P_{S,\alpha\,r}(\underline{A}_i)];\ \mu(P_{S,\alpha}(\underline{A}_i)) = \alpha\ \forall\ \alpha \in (0;1]\} \tag{6}$$

The α-level sets $P_{S,\alpha}(\underline{A}_i)$ are determined from

$$P_{S,\alpha\,r}(\underline{A}_i) = P(\underline{X}_\alpha \cap \underline{A}_i \neq \varnothing \wedge \underline{X}_\alpha \not\subseteq \underline{A}_i) \tag{7}$$

i.e., the following holds

$$P_{S,\alpha\,r}(\underline{A}_i) = P_{\alpha\,r}(\underline{A}_i) - P_{\alpha\,l}(\underline{A}_i) \tag{8}$$

Ordinary random variables $\underline{X}$ do not possess probability shadows. A Fuzzy random variable may be considered to be an "ordinary random variable extended by the probability shadow".

The properties of the fuzzy probability $\tilde{P}(\underline{A}_i)$ result from the properties of the uncertain measure probability, under consideration of all σ-algebras $S_\alpha(\underline{\mathbf{X}})$ contained in $\tilde{S}(\underline{\mathbf{X}})$. For example, a complementarity relationship may be derived for $\tilde{P}(\underline{A}_i)$

$$\tilde{P}(\underline{A}_i) = 1 - \tilde{P}(\underline{A}_i^C) \tag{9}$$

2.3 Fuzzy probability distributions

The fuzzy probability $\tilde{P}(\underline{A}_i)$ of $\underline{A}_i$ may be computed for each arbitrary set $\underline{A}_i \in S(\underline{\mathbf{X}})$. If (as a special system of sets $S(\underline{\mathbf{X}})$) the system $S_0(\mathbb{R}^n)$ of the unbounded sets

$$\underline{A}_i = \left\{ \underline{t} = (t_1; ...; t_k; ...; t_n) \mid \underline{x} = \underline{x}_i ; \underline{x}, \underline{t} \in \mathbb{R}^n ; t_k < x_k ; k = 1, ..., n \right\} \tag{10}$$

is chosen, the concept of the probability distribution function may then be applied to fuzzy random variables.

The *fuzzy probability distribution function* $\tilde{F}(\underline{x})$ of the fuzzy random variable $\tilde{\underline{X}}$ over $\underline{\mathbf{X}} = \mathbb{R}^n$ is the fuzzy probability $\tilde{P}(\underline{A}_i)$ of $\underline{A}_i$ according to Eqn. (3) for all $\underline{x}_i \in \underline{\mathbf{X}}$. The fuzzy functional values

$$\tilde{F}(\underline{x}) = \left\{ \left(F_\alpha(\underline{x}) ; \mu(F_\alpha(\underline{x})) \right) \mid F_\alpha(\underline{x}) = [F_{\alpha l}(\underline{x}) ; F_{\alpha r}(\underline{x})] ; \mu(F_\alpha(\underline{x})) = \alpha \, \forall \, \alpha \in (0;1] \right\} \tag{11}$$

are defined for all α-levels by

$$F_{\alpha l}\left(\underline{x} = (x_1; ...; x_n)\right) = 1 - \max_j P\left(\underline{X}_{j,\alpha} = \underline{t} = (t_1; ...; t_n) \mid \underline{x}, \underline{t} \in \underline{\mathbf{X}} = \mathbb{R}^n ; \exists\, t_k \geq x_k ; 1 \leq k \leq n \right) \tag{12}$$

$$F_{\alpha r}\left(\underline{x} = (x_1; ...; x_n)\right) = \max_j P\left(\underline{X}_{j,\alpha} = \underline{t} = (t_1; ...; t_n) \mid \underline{x}, \underline{t} \in \underline{\mathbf{X}} = \mathbb{R}^n ; t_k < x_k ; k = 1, ..., n \right) \tag{13}$$

The fuzzy probability distribution function $\tilde{F}(\underline{x})$ *of* $\tilde{\underline{X}}$ *is the set of the probability distribution functions* $F_j(\underline{x})$ *of all originals* $\underline{X}_j$ *of* $\tilde{\underline{X}}$ *with the membership values* $\mu(F_j(\underline{x}))$, see Fig. 5.

The application of Eqn. (6) yields the *probability distribution shadow*

$$\tilde{F}_S(\underline{x}) = \left\{ \left(F_{S,\alpha}(\underline{x}) ; \mu(F_{S,\alpha}(\underline{x})) \right) \mid F_{S,\alpha}(\underline{x}) = [0 ; F_{S,\alpha r}(\underline{x})] ; \mu(F_{S,\alpha}(\underline{x})) = \alpha \, \forall \, \alpha \in (0;1] \right\} \tag{14}$$

$$F_{S,\alpha r}(\underline{x}) = F_{\alpha r}(\underline{x}) - F_{\alpha l}(\underline{x}) \tag{15}$$

of the fuzzy random variable $\tilde{\underline{X}}$, which exclusively describes the fuzziness in $\tilde{F}(\underline{x})$.

The *fuzzy probability density function* $\tilde{f}(\underline{t})$ (or $\tilde{f}(\underline{x})$), see Fig. 5, is a function belonging to $\tilde{F}(\underline{x})$, which, in the continuous case (in relation to randomness), is integrable for each original $\underline{X}_j$ of $\tilde{\underline{X}}$. For $\underline{t} = (t_1; ...; t_n) \in \mathbf{\underline{X}}$, the following holds

$$F_j(\underline{x}) = \int_{t_1=-\infty}^{t_1=x_1} \dots \int_{t_k=-\infty}^{t_k=x_k} \dots \int_{t_n=-\infty}^{t_n=x_n} f_j(\underline{t})\, d\underline{t} \tag{16}$$

For discrete fuzzy random variables (in relation to their randomness) the integral term reduces to a sum.

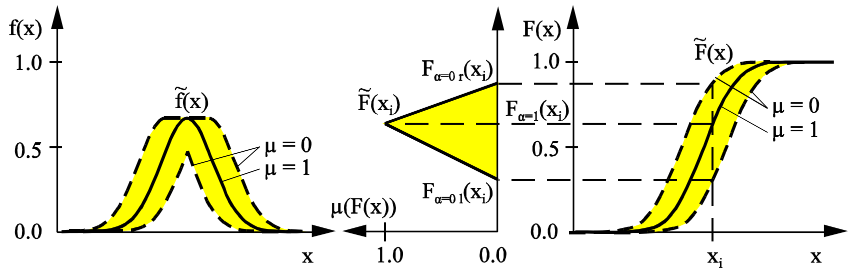

Fig. 5 Fuzzy probability density function $\tilde{f}(x)$ and fuzzy probability distribution function $\tilde{F}(x)$ of a one-dimensional continuous fuzzy random variable

2.4 Parameters of fuzzy random variables

Fuzzy probability is based on the special *objective* uncertain measure probability, which must be applicable to each original $\underline{X}_j$ of a fuzzy random variable $\tilde{\underline{X}}$. One original of $\tilde{\underline{X}}$ is only able to describe uncertainty with the characteristic *randomness*. Uncertainty with the characteristic *fuzziness* is accounted for by considering different originals of $\tilde{\underline{X}}$. Based on observed, random realizations of $\tilde{\underline{X}}$ (or their subjective assessment) the originals $\underline{X}_j$ are determined. The computed originals $\underline{X}_j$ are assessed using membership values. The fuzzy random variable $\tilde{\underline{X}}$ is obtained as the fuzzy set of all originals $\underline{X}_j$. The corresponding fuzzy probability distribution function $\tilde{F}(\underline{x})$ maps the fundamental set $\mathbf{\underline{X}}$ on the interval [0; 1]. The functional values of the fuzzy probability distribution function $\tilde{F}(\underline{x})$ are fuzzy numbers.

The distribution type and parameters of the fuzzy random variables must be determined by evaluating all originals $\underline{X}_j$ of $\tilde{\underline{X}}$; these are obtained in the form of fuzzy parameters $\tilde{p}_t(\tilde{\underline{X}})$.

Ther *fuzzy parameter* $\tilde{p}_t(\tilde{\underline{X}})$ of the fuzzy random variable $\tilde{\underline{X}}$ is the fuzzy set of the parameters $p_t(\underline{X}_j)$ of all originals $\underline{X}_j$ with the membership values $\mu(p_t(\underline{X}_j))$

$$\tilde{p}_t(\tilde{\underline{X}}) = \left\{ \left(p_t(\underline{X}_j); \mu(p_t(\underline{X}_j)) \right) \mid \underline{X}_j \in \tilde{\underline{X}};\ \mu(p_t(\underline{X}_j)) = \mu(\underline{X}_j)\ \forall j \right\} \tag{17}$$

By applying α-discretization, $\tilde{p}_t(\underline{\tilde{X}})$ is defined by

$$\tilde{p}_t(\underline{\tilde{X}}) = \left\{ \left(p_{t,\alpha}(\underline{\tilde{X}}); \mu(p_{t,\alpha}(\underline{\tilde{X}})) \right) \mid \mu(p_{t,\alpha}(\underline{\tilde{X}})) = \alpha \forall \alpha \in (0;1] \right\} \quad (18)$$

$$p_{t,\alpha}(\underline{\tilde{X}}) = [p_{t,\alpha l}(\underline{\tilde{X}}); p_{t,\alpha r}(\underline{\tilde{X}})] \quad (19)$$

$$p_{t,\alpha l}(\underline{\tilde{X}}) = \min_j [p_t(\underline{X}_j) \mid \underline{X}_j \in \underline{\tilde{X}}] \quad (20)$$

$$p_{t,\alpha r}(\underline{\tilde{X}}) = \max_j [p_t(\underline{X}_j) \mid \underline{X}_j \in \underline{\tilde{X}}] \quad (21)$$

3 Generation of fuzzy probability distribution functions

The quantification of uncertain structural parameters as fuzzy random variables requires the application of statistical methods and the consideration of expert knowledge. For this purpose the uncertainty is separated into fuzziness and randomness. Statistical algorithms are applied to describe randomness, whereas the fuzziness component is handled by means of fuzzification algorithms. The aim is to determine the fuzzy parameters $\tilde{p}_t(\underline{\tilde{X}})$ of the fuzzy random variables. Fuzzy probability distribution functions and fuzzy probability density functions with a fuzzy functional type and fuzzy functional parameters are introduced. These fuzzy functions are modeled as a bunch of functions; the computed $\tilde{p}_t(\underline{\tilde{X}})$ constitute these bunch parameters.

The cause of fuzzy randomness determines the principal approach for computing the fuzzy parameters $\tilde{p}_t(\underline{\tilde{X}})$. Owing to the subjective nature of fuzzification, the algorithms for formulating fuzzy probability distribution functions must be matched to the particular problem concerned.

3.1 Small sample size

A sample of limited size is available. Further information on the statistical properties of the universe are not available. The sample elements possess uncertainty with the characteristic randomness. The information provided by the sample, however, is not sufficient to describe an ordinary random variable without some element of doubt; further uncertainty exists. This uncertainty is identified as informal uncertainty with the characteristic fuzziness. When fuzzifying informal uncertainty, however, statistical aids may be implemented. Depending on the available information, either a parametric or a non-parametric estimation problem is formulated.

If, for example, the type of distribution is known to a sufficient degree of certainty, the estimation problem concerned is a parametric one. The sample statistics applied in statistical methods yield more or less good estimates of values for the sought parameters. In order to take account of the uncertainty of the estimator, confidence intervals may be determined for the estimator concerned. The probabilistic postulations of confidence intervals used in statistical methods only serve here as a guideline for the fuzzification of $\tilde{p}_t(\underline{\tilde{X}})$. This only provides information, however, concerning the region in which a parameter "may possibly lie". This permits the derivation of a fuzzification proposal for $\tilde{p}_t(\underline{\tilde{X}})$, which serves as an initial draft of the membership function $\mu(p_t(\underline{X}))$.

Expert knowledge is always drawn upon when determining the fuzzy parameters $\tilde{p}_t(\underline{\tilde{X}})$. Subjective information is taken into consideration, especially when
- selecting the estimator,
- constructing the confidence intervals (type and level) and
- subsequently modifying the initial draft of the membership functions of the fuzzy parameters.

3.2 Fuzzy samples – unknown, non-constant reproduction conditions

For the case of a sufficiently large sample size and constant reproduction conditions a sample exclusively possesses uncertainty with the characteristic randomness. If the reproduction conditions are not constant, however, the uncertainty characteristics alter. The additional uncertainty may be accounted for as fuzziness.

If the reproduction conditions are not known, the observed realizations then possess informal uncertainty. If the uncertainty of each realization is described by means of fuzzy numbers, it is possible to formulate a fuzzy random variable from the data material. An observed fuzzy realization is assigned to each elementary event.

Sample elements with informal uncertainty may, for example, arise when determining the compressive strength of concrete. The production of test samples may be carried out by different persons with differing degrees of care. The hardening conditions for concrete on site and at different sample storage locations may differ as a result of different ambient conditions (e.g. temperature and humidity). Different measuring equipment (also of a different type) may be used for testing compressive strength, whereby each device yields different errors of measurement. The staff responsible for testing may conduct measurements with individual degrees of conscientiousness. The sample elements thus possess uncertainty with the characteristic fuzziness; it is thus appropriate to model compressive strength as a fuzzy random variable.

The originals and parameters of the fuzzy random variables must be determined or estimated on the basis of the fuzzy realizations. Algorithms employed in mathematical statistics are applied as a mapping operator for this purpose. Each fuzzy realization represents an input value for the mapping operator.

3.3 Fuzzy samples – known, non-constant reproduction conditions

In contrast to Section 3.2, the causes of non-constant reproduction conditions are now known in detail. A knowledge of these causes serves to separate fuzziness and randomness in the statistical data material.

This separation process presupposes that the causes of non-constant reproduction conditions may be characterized by certain attributes. Observed realizations with the same attributes are allocated to a particular group. Each group with the same attributes is treated as a random sample and evaluated using statistical methods. The statistical evaluation of all groups yields the set $\underline{S}$ of statistical prognoses. A subset of the universe is assigned to each element of $\underline{S}$. The set $\underline{S}$ is uncertain and characterizes the fuzziness of the universe. The fuzzy set $\tilde{\underline{S}}$ thus describes the set of ordinary random variables contained in the observed realizations. These ordinary random variables are originals of the sought fuzzy random variable, and the fuzzy set $\tilde{\underline{S}}$ assigns membership values to the parameters of the originals. The membership functions of the statistical parameters may be constructed with the aid of histograms. An alternative possibility is the direct fuzzification of the probability distribution function curve.

The histograms of statistical parameters are specific to a particular problem and serve as design aids for defining the membership functions. The variable to be fuzzified (statistical parameter) is plotted along the abscissa, previously subdivided into subsets. A particular value, assigned to a subset, belongs to each group of realizations. This value represents a sample element of the histogram. The number of sample elements assigned to a subset is plotted along the ordinate of the histogram. The evaluation of the histogram in order to formulate the membership function is carried out under consideration of expert knowledge; variants result from the choice of different subset widths.

When directly fuzzifying the curve of the probability distribution function, an empirical probability distribution function $F_i^e(x)$ is developed for each group. The evaluation of all groups yields a bunch of empirical (i.e. discrete) distribution functions. The functional values $F^e(x)$ are plotted at (discrete) points x in histograms and directly fuzzified. For different membership levels an attempt is made to determine the originals of the fuzzy random variable which limit these levels. This approximation may be carried out, e.g. using the method of least square errors. All enclosed originals obtained in this way together describe the sought fuzzy probability distribution.

4 Fuzzy probabilistic safety assessment

The aim of fuzzy probabilistic safety assessment is to determine and assess the fuzzy safety level. Fuzzy random variables, ordinary random variables and fuzzy variables may thereby be accounted for simultaneously. The uncertainty of the input data and the (computational) model is apparent in the results of the safety assessment, i.e. in the fuzzy failure probability and the fuzzy reliability index. The fuzziness of the computed safety level characterizes the new quality of the safety assessment compared with probabilistic methods.

The fuzzy probabilistic safety assessment requires a stochastic fundamental solution. In principle, any probabilistic algorithm may be used for this purpose. By way of example, the First Order Reliability Method (FORM) is chosen here and extended to yield the *Fuzzy First Order Reliability Method (FFORM)*.

4.1 Original space of the fuzzy probabilistic basic variables

The original space is constructed using basic variables. These are specified as fuzzy random variables and enter the safety assessment as fuzzy probabilistic basic variables together with fuzzy variables, which are model parameters of the uncertain structural model. Ordinary random variables (as a special case of fuzzy random variables) may at the same time be accounted for as probabilistic basic variables. In the case of n fuzzy probabilistic basic variables $\tilde{X}_i$ an original space (x-space) results with n axes x_i.

The fuzzy probability density functions $\tilde{f}(x_i)$ of the $\tilde{X}_i$ are lumped together in the original space to form the *fuzzy joint probability density function* $\tilde{f}(\underline{x})$; $\tilde{f}(\underline{x})$ includes all combinations of the originals of the fuzzy probabilistic basic variables. Each combination yields one crisp joint probability density function, and a functional value $\tilde{f}(\underline{x})$ is assigned to each point in the x-space. The membership values μ of the $\tilde{f}(\underline{x})$ are determined using the max-min operator of the extension principle [1]. *The fuzzy joint probability density function* $\tilde{f}(\underline{x})$ *is a fuzzy set, which is constructed from the set of all originals and the membership values.*

Using the fuzzy numbers $\tilde{p}_t(\tilde{X}_i)$ for the parameters of the fuzzy random variables $\tilde{X}_i$, the fuzzy joint probability density function is the result of the mapping

$$\left\{\tilde{p}_t(\tilde{X}_i)\,;\, i = 1,...,n\,;\, t = 1,...,r_i\right\} \rightarrow \tilde{f}(\underline{x}) \tag{22}$$

Ordinary random variables are treated as fuzzy probabilistic basic variables with only one original. For the special case that all basic variables possess only one original, the fuzzy joint probability density function $\tilde{f}(\underline{x})$ reduces to the ordinary joint probability density function $f(\underline{x})$.

The limit state surface is specified by the computational model. Uncertain computational models with fuzzy model parameters result in a *fuzzy limit state surface* $\tilde{g}(\underline{x}) = 0$ in the original space of the basic variables. The space of the fuzzy probabilistic basic variables is subdivided into a fuzzy survival region $\tilde{\underline{X}}_s$ and a fuzzy failure region $\tilde{\underline{X}}_f$ by the fuzzy limit state surface. The fuzzy function $\tilde{g}(\underline{x}) = 0$ may be expressed in the form

$$\left(\tilde{g}(\underline{x}) = 0\right) = \left\{\left(g(\underline{x}) = 0; \mu(g(\underline{x}) = 0)\right) \mid \underline{x} \in \underline{\mathbf{X}}\right\} \tag{23}$$

The $g(\underline{x}) = 0$ form a bunch of functions with the membership values $\mu(g(\underline{x}) = 0)$; these are elements of the fuzzy set $\tilde{g}(\underline{x}) = 0$ and represent crisp limit state surfaces.

In order to compute the elements $g(\underline{x}) = 0$ it is necessary to discretize the fuzzy model parameters, i.e. selection of an α-level and selection of elements from the α-level sets. By this means, possible values of the fuzzy model parameters are defined. These values serve as input data to a (non-linear) analysis algorithm with which the crisp limit state surface $g(\underline{x}) = 0$ may be computed. The respective analysis algorithm is referred to as the deterministic fundamental solution. The quality of the deterministic fundamental solution has a decisive influence on the results of the safety assessment; thus the system behavior of the structure has to be realistically numerically simulated.

The assessment of the points $\underline{x}$ in the space of the fuzzy probabilistic basic variables regarding failure or survival is carried out using membership functions, see Fig. 6. For the fuzzy survival region $\tilde{\underline{X}}_s$ the following holds

$$\mu(\underline{x}_s) = \mu(g(\underline{x})>0) = \begin{cases} 1 & \forall\, \underline{x} \mid g(\underline{x})_{\alpha=1} \geq 0 \\ \mu(g(\underline{x})=0) & \text{otherwise} \end{cases} \tag{24}$$

The membership function of the fuzzy failure region $\tilde{\underline{X}}_f$ is given by

$$\mu(\underline{x}_f) = \mu(g(\underline{x}) \leq 0) = \begin{cases} 1 & \forall\, \underline{x} \mid g(\underline{x})_{\alpha=1} \leq 0 \\ \mu(g(\underline{x})=0) & \text{otherwise} \end{cases} \tag{25}$$

The membership values $\mu(\underline{x}_s)$ and $\mu(\underline{x}_f)$ assess the postulations $\underline{x} \in \tilde{\underline{X}}_s$ and $\underline{x} \in \tilde{\underline{X}}_f$ for all points $\underline{x} \in \underline{\mathbf{X}}$. The survival and failure regions overlap in the region of the fuzzy limit state surface $\tilde{g}(\underline{x}) = 0$.

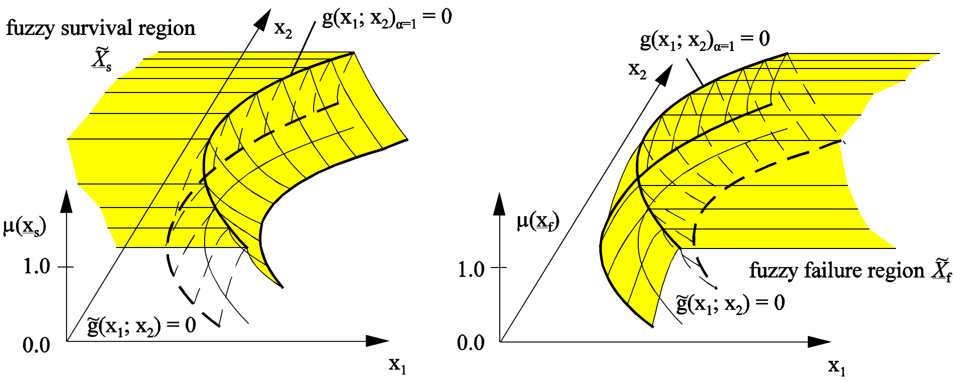

Fig. 6 Fuzzy limit state surface $\tilde{g}(\underline{x}) = 0$, fuzzy survival region $\tilde{\underline{X}}_s$ and fuzzy failure region $\tilde{\underline{X}}_f$

In the original space of the basic variables the fuzzy joint probability density function $\tilde{f}(\underline{x})$ is plotted together with the fuzzy limit state surface $\tilde{g}(\underline{x}) = 0$, see Fig. 7. The sought fuzzy failure probability $\tilde{P}_f$ is obtained by integrating $\tilde{f}(\underline{x})$ over the failure region $\tilde{\underline{X}}_f$ with $\tilde{g}(\underline{x}) \leq 0$. Analogous to the first order reliability method, this integration is replaced by the transformation of the problem into the standard normal space (y-space) and the determination of the fuzzy design point and the fuzzy reliability index.

In general, a fuzzy design point $\tilde{\underline{x}}_B$ is obtained in the x-space, which exhibits fuzziness in the direction of $\tilde{g}(\underline{x}) = 0$ (coordinate s) as well as in the direction at right angles to the latter (coordinate t), see Fig. 7. The fuzziness of $\tilde{\underline{x}}_B$ is determined by the fuzzy parameters $\tilde{p}_t(\tilde{X}_i)$

(contained in $\tilde{f}(\underline{x})$) and the fuzzy model parameters $\tilde{m}_j$ (contained in $\tilde{g}(\underline{x}) = 0$). The fuzzy design point $\tilde{\underline{x}}_B$ is the result of the mapping

$$\left\{ \tilde{p}_t(\tilde{X}_i)\,;\, \tilde{m}_j\,;\, i = 1, ..., n\,;\, t = 1, ..., r_i\,;\, j = 1, ..., q \right\} \rightarrow \tilde{\underline{x}}_B \tag{26}$$

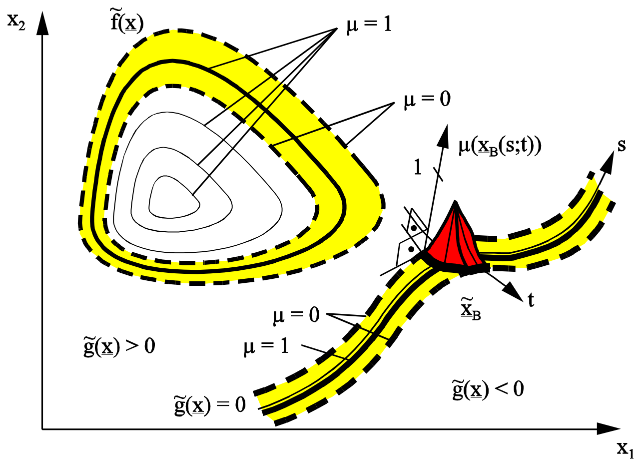

Fig. 7 Fuzzy joint probability density function $\tilde{f}(\underline{x})$, fuzzy limit state surface $\tilde{g}(\underline{x}) = 0$ and fuzzy design point $\tilde{\underline{x}}_B$ with data and model uncertainty

Each combination of elements of the fuzzy model parameters $\tilde{m}_j$ yields precisely one element of the fuzzy limit state surface $\tilde{g}(\underline{x}) = 0$, and each combination of elements of the fuzzy parameters $\tilde{p}_t(\tilde{X}_i)$ yields precisely one original of the fuzzy joint probability density function $\tilde{f}(\underline{x})$. *The evaluation of all combinations of elements of* $\tilde{g}(\underline{x}) = 0$ *and the originals of* $\tilde{f}(\underline{x})$ *yields the elements of the fuzzy design point* $\tilde{\underline{x}}_B$ *with the membership values* $\mu(\underline{x}_B)$.

4.2 Standard normal space and fuzzy reliability index

For FFORM it is necessary to transform the fuzzy random variables $\tilde{X}_i$ into standard normalized random variables Y_i. The fuzzy random variable $\tilde{X}_i$ has the fuzzy probability distribution function $\tilde{F}(x)$; the standard normal distribution $\Phi^{NN}(y)$ for the new random variable Y_i is defined as a crisp probability distribution function, however. For this reason, the transformation relationship between $\tilde{X}_i$ and Y_i contains uncertainty with the characteristic fuzziness; the uncertain transformation of crisp values $x \in \mathbf{X}$ leads to fuzzy variables $\tilde{y} \in \mathbf{Y}$, see Fig. 8. The transformation $x \tilde{\rightarrow} \tilde{y}$ is realized with the aid of the (fuzzy) probability distribution functions $\tilde{F}(x)$ and $\Phi^{NN}(y)$ on an original-to-original basis

$$\tilde{y} = \Phi^{NN^{-1}}(\tilde{F}(x)) \tag{27}$$

The transformation of limit state points $\underline{x}$ or $\tilde{\underline{x}}$ according to Eqn. (27) always leads to fuzzy limit state points $\tilde{\underline{y}}$ in the standard normal space. For this reason, the limit state surface in the y-space is always obtained as a fuzzy limit state surface $\tilde{h}(\underline{y}) = 0$, see Fig. 9.

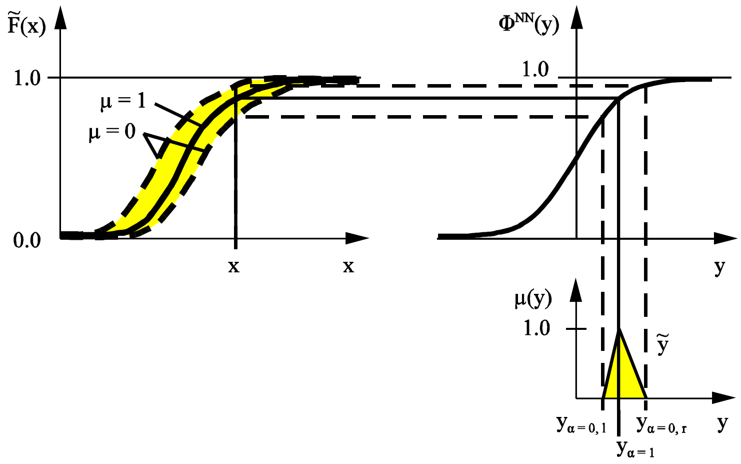

Fig. 8 Transformation of fuzzy random variables into standard normalized random variables

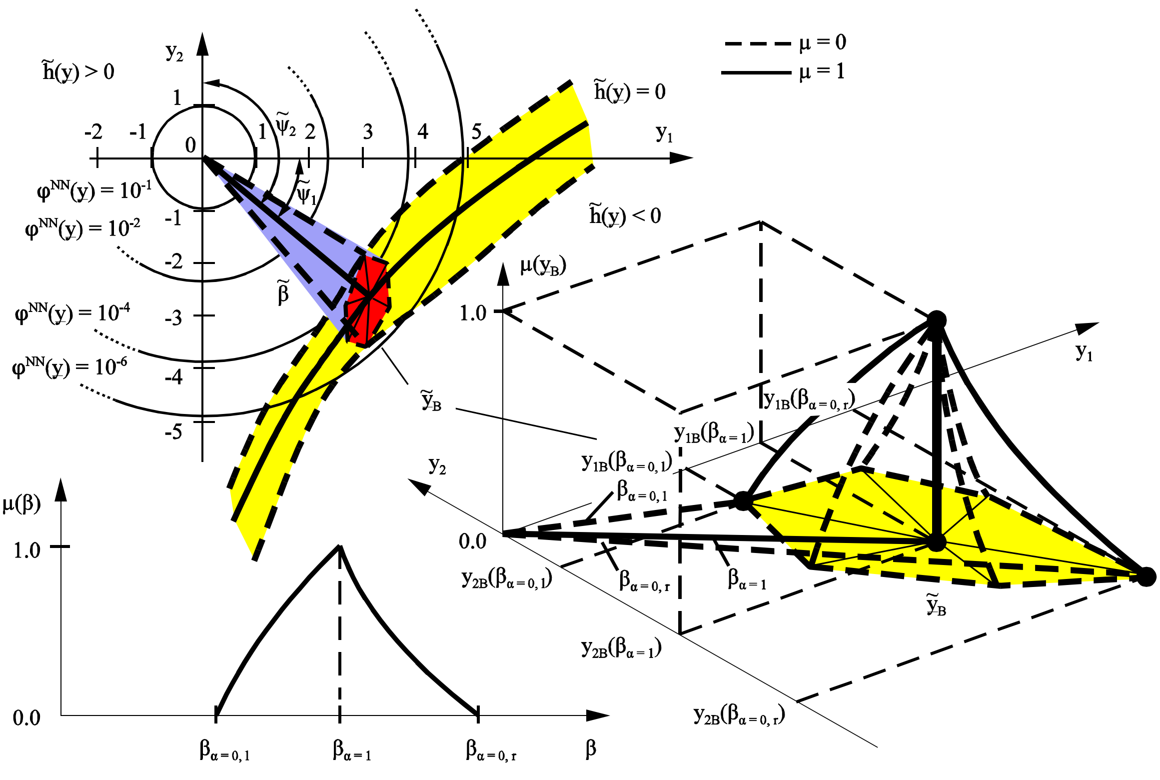

Fig. 9 Fuzzy limit state surface $\tilde{h}(\underline{y}) = 0$, fuzzy design point $\tilde{\underline{y}}_B$ and fuzzy reliability index $\tilde{\beta}$

In the transformation from the x-space into the y-space the uncertainty of the structural parameters (data and model uncertainty) is separated according to the characteristics fuzziness and randomness. The joint standard normal probability distribution exclusively describes uncertainty with the characteristic randomness whereas the fuzzy limit state surface $\tilde{h}(\underline{y}) = 0$ only possesses uncertainty with the characteristic fuzziness.

An evaluation of the fuzzy limit state surface $\tilde{h}(\underline{y}) = 0$ yields the fuzzy design point $\tilde{\underline{y}}_B$ and the fuzzy reliability index $\tilde{\beta}$, see Fig. 9. The fuzzy parameters $\tilde{p}_t(\tilde{X}_i)$ of the fuzzy probabilistic basic variables and the fuzzy model parameters $\tilde{m}_j$ are mapped onto the fuzzy design point $\tilde{\underline{y}}_B$ and the fuzzy reliability index $\tilde{\beta}$

$$\left\{ \tilde{p}_t(\tilde{X}_i) ; \tilde{m}_j ; i = 1,...,n ; t = 1,...,r_i ; j = 1,...,q \right\} \rightarrow \tilde{\underline{y}}_B ; \tilde{\beta} \tag{28}$$

5 Example

It is proposed to determine the structural reliability of the steel girder shown in Fig. 10. System failure is considered according to first order plastic joint theory. On attaining the system ultimate load, the cross-section at point k is completely plasticized; Fig. 10 shows the corresponding failure mechanism.

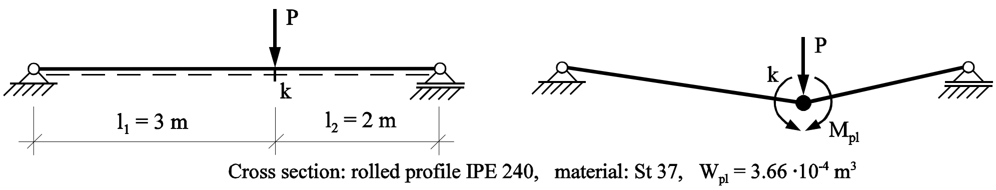

Fig. 10 Steel girder; static system with loading; governing failure mechanism

The load P and the yield stress f_y are modeled as fuzzy random variables $\tilde{X}_1$ and $\tilde{X}_2$, respectively. A fuzzy extreme value distribution of Ex-Max Type I is chosen for P, with the fuzzy mean value $\tilde{m}_{X_1}$ and the fuzzy standard deviation $\tilde{\sigma}_{X_1}$ according to Fig. 11. The yield stress f_y is assumed to follow a logarithmic normal distribution. The minimum value of $x_{0,2} = 19.9 \cdot 10^4$ kN/m² and the mean value of $m_{X_2} = 28.8 \cdot 10^4$ kN/m² are prescribed. The standard deviation enters the safety assessment as a fuzzy triangular number $\tilde{\sigma}_{X_2}$, see Fig. 11.

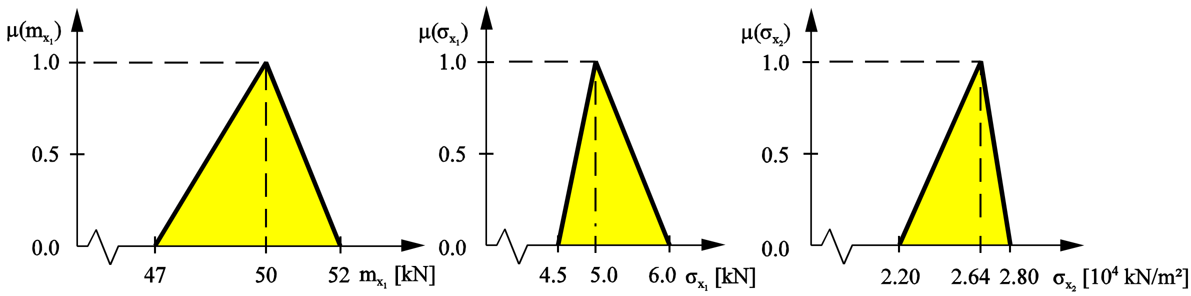

Fig. 11 Fuzzy parameters $\tilde{m}_{X_1}$, $\tilde{\sigma}_{X_1}$ and $\tilde{\sigma}_{X_2}$

The fuzzy joint probability density function $\tilde{f}(x_1;x_2)$ (Fig.12) and the crisp linear limit state surface

$$g(\underline{x}) = g(x_1;x_2) = x_2 - \frac{l_1 \cdot l_2}{W_{pl}(l_1 + l_2)} \cdot x_1 = 0 \tag{29}$$

are obtained in the x-space. FFORM is applied to compute the fuzzy reliability index $\tilde{\beta}$ in the y-space, see Fig. 13. The back-transformed fuzzy design point $\underline{\tilde{x}}_B$ is plotted in the x-space in Fig. 12.

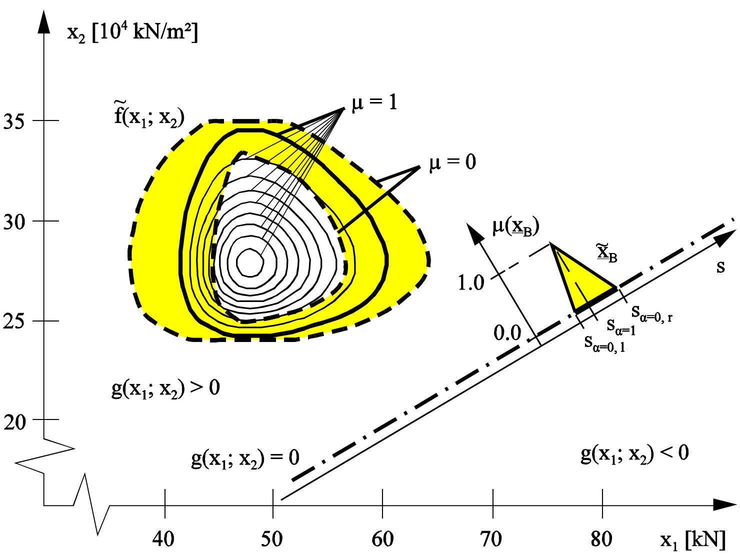

Fig. 12 Fuzzy joint probability density function $\tilde{f}(x_1;x_2)$, crisp limit state surface $g(x_1; x_2) = 0$ and fuzzy design point $\underline{\tilde{x}}_B$

The safety verification is carried out by comparing the fuzzy reliability index $\tilde{\beta}$ with required values *req*_β. If the safety level to be complied with is specified as *req*_β = 3.8, then the safety verification $\tilde{\beta} \geq$ *erf*_β is only *partially fulfilled; a subjective assessment* is necessary, see Fig. 13. The measure values μ_1 and μ_2 are obtained from the fuzzy sets $\tilde{\beta}_1$ and $\tilde{\beta}_2$ shown in Fig. 13. The safety verification is assessed as being fulfilled with $\mu_1 = 0.5$, but not fulfilled with $\mu_2 = 1.0$. A decision as to whether the safety verification may be considered to be fulfilled must be made on the basis of expert knowledge.

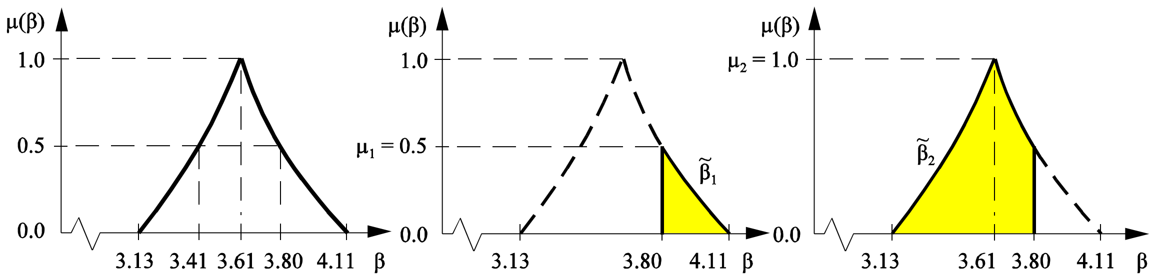

Fig. 13 Fuzzy reliability index $\tilde{\beta}$ and safety verification

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